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# Planar graph bipartization in linear time<sup>☆</sup>

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## Abstract

For each constant  $k$ , we present a linear time algorithm that, given a planar graph  $G$ , either finds a minimum odd cycle vertex transversal in  $G$  or guarantees that there is no transversal of size at most  $k$ .

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## 1. Introduction

An *odd cycle transversal* (or *cover*) is a subset of the vertices of a graph  $G$  that hits all the odd cycles in  $G$ . Clearly the deletion of such a vertex set leaves a bipartite graph. Thus the problem of finding an odd cycle transversal of minimum cardinality is just the classical *graph bipartization problem*. Whilst this problem is NP-hard, it was recently shown [12] that an  $O(n^2)$  time algorithm does exist when the size of an optimal solution is constant. This result is of particular interest given that in many practical examples, for example, in computational biology [13], the transversals are typically small.

In this paper, we consider the restriction of the graph bipartization problem to planar graphs. As the vertex cover problem in planar graphs can be reduced to it, the restricted problem is still NP-hard. This and other related vertex and edge deletion problems in planar graphs have been extensively studied both structurally and algorithmically (see, for example, [10,5,8]). Here we give a linear time algorithm for instances with constant sized optimal solutions. The graph properties of consequence in this problem are very different for planar graphs than for general graphs. By exploiting these properties, we develop an algorithm quite unlike that of [12].

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We consider an embedding of the planar graph  $G$ . The *parity* of a face of  $G$  is defined as the parity of the edge set of its boundary, counting bridges twice. The crucial observation is that the parity of a cycle in  $G$  is equal mod 2 to the sum of the parities of the faces within it. In particular, it follows from the crucial observation that  $G$  is bipartite if and only if all its faces are even.

When a vertex  $v$  is deleted from  $G$ , all the faces incident to  $v$  are merged together in a new face  $F$ . The other faces are unchanged. We denote the new face by a capital letter to stress the fact that it determines a set of faces of  $G$ , namely, the faces of  $G$  included in it. Note that the parity of the new face  $F$  equals the sum mod 2 of the parities of the faces of  $G$  it contains. Let now  $W$  denote any set of vertices in  $G$ . By deleting from  $G$  the vertices of  $W$  one after the other in some order, we see that each face of  $G - W$  corresponds to a set of faces of  $G$ . This set is a singleton if the corresponding face is a face of  $G$  that survived in  $G - W$ . Furthermore, a face of  $G - W$  is odd precisely if it contains an odd number of odd faces of  $G$ . Because a planar graph is bipartite if and only if all its faces are even, we obtain our

**Key fact.** *A set  $W$  of vertices is an odd cycle transversal of  $G$  precisely if every face of  $G - W$  contains an even number of odd faces of  $G$ .*

We remind readers that for a given embedding of  $G$ , the *face-vertex incidence graph* of  $G$  is the bipartite graph  $G^+$  on the vertices and faces of  $G$  whose edges are the pairs  $fv$ , where  $f$  is a face of  $G$  and  $v$  is a vertex of  $G$  incident to  $f$ . In the next paragraph, we state a useful reformulation of the Key fact in terms of  $T$ -joins in the face-vertex incidence graph. For the graph bipartization problem using edge deletions, Hadlock [6] considered a similar relationship between odd cycle (edge) transversals and  $T$ -joins in the dual graph. He used this to give a polynomial time algorithm for the maximum cut problem in planar graphs. Because the maximum cut problem in a planar graph is equivalent to the minimum  $T$ -join problem in the dual graph, his algorithm is, in fact, a polynomial-time algorithm for the bipartization problem by edge deletion.

Consider any graph  $H$  and set of vertices  $T$  in  $H$ . A  $T$ -join in  $H$  is a set of edges  $J$  such that  $T$  equals the set of odd degree vertices in the subgraph of  $H$  determined by  $J$ . There exists a  $T$ -join in  $H$  if and only if each connected component of  $H$  contains an even number of vertices of  $T$ . In particular, if  $H$  has a  $T$ -join then  $|T|$  is even. Now let  $T$  be the set of odd faces of the planar graph  $G$ . So  $T$  is an even set of vertices in the face-vertex incidence graph  $G^+$ . Letting  $F(G)$  denote the set of faces of  $G$ , the correspondence between odd cycle transversals is as follows.

**Lemma 1.1.** *A subset  $W$  of  $V(G)$  is an odd cycle transversal of  $G$  if and only if the subgraph of  $G^+$  induced on  $W \cup F(G)$  contains a  $T$ -join, that is, every connected component of the subgraph has an even number of vertices of  $T$ .*

**Proof.** The lemma is equivalent to the Key fact because deleting a vertex  $v$  from  $G$  corresponds to contracting all the edges incident to  $v$  in the face-vertex incidence graph  $G^+$ .  $\square$

Consider an inclusionwise minimal odd cycle transversal  $W$  of  $G$ . By the above lemma, i.e., by the Key fact, there is a  $T$ -join  $J$  in  $G^+$  covering each vertex of  $W$  and no vertex of  $G - W$ . Without loss of generality, we can assume that  $J$  is inclusionwise minimal. Then  $J$  is a forest and every leaf of  $J$  is in  $T$ . Hence, Lemma 1.1 is useful because it enables us to visualize odd cycle transversals of  $G$  as forests in the face-vertex incidence graph  $G^+$  such that each tree of the forest contains an even number of vertices of  $T$ . Furthermore, note that some vertices of  $T$  can be internal nodes of  $J$ . For every vertex  $v$  of  $W$ , there are two internally disjoint paths in  $J$  between  $v$  and  $T$ . So, letting  $d_{\min}(x)$  be the minimum length of a path from  $x$  to an odd face in the face-vertex incidence graph, we see that the Key fact implies:

**Corollary 1.2.** *No vertex  $v$  is in a minimal odd cycle transversal of size less than  $d_{\min}(v)$ .*

Thus, letting  $G'$  be the subgraph of  $G$  induced by  $\{v \in V(G) \mid d_{\min}(v) > k\}$  we see that if  $G$  has an odd cycle transversal of size at most  $k$  then  $G'$  must be bipartite. That is,  $V - V(G')$  is an odd cycle transversal. So applying the Key fact to the embedding of  $G'$  which appears as a sub-embedding of our embedding of  $G$ , we obtain:

**Corollary 1.3.** *If  $G$  has an odd cycle transversal of size at most  $k$  then every face  $F$  of  $G'$  contains an even number of odd faces of  $G$ .*

We note further that the boundary,  $bd(F)$ , of every face  $F$  of  $G'$  is disjoint from the boundaries of the odd faces of  $G$  within it by the definition of  $G'$  (except for the trivial case  $k = 0$ ). Thus we have:

**Observation 1.4.** *If  $G$  has an odd cycle transversal of size at most  $k$  then there are at most  $k$  faces of  $G'$  which contain an odd face of  $G$ .*

For some  $r \leq k$ , we let  $\{F_1, \dots, F_r\}$  be the set of faces of  $G'$  containing an odd face of  $G$  and let  $G_i = G \cap (F_i \cup bd(F_i))$ . Applying Corollary 1.2 again, it is easy to show:

**Corollary 1.5.** *If  $G$  has an odd cycle transversal of order at most  $k$  then  $W$  is a minimum odd cycle transversal of  $G$  precisely if  $W_i = W \cap G_i$  is a minimum odd cycle transversal of  $G_i$  for every  $i$  between 1 and  $r$ .*

**Proof.** Consider an odd cycle transversal  $W$  of  $G$  of order at most  $k$ . By Corollary 1,  $W_i$  is disjoint from  $bd(F_i)$ , and each face of  $G - W$  which is not a face of  $G'$  is a face of  $G_i - W_i$  for some  $i$ . Thus, applying the Key fact to  $G - W$  and  $G_i - W_i$  for each  $i$  we see that  $W$  is an odd cycle transversal of  $G$  if and only if  $W_i$  is an odd cycle transversal of  $G_i$  for each  $i$ . Since  $W_i$  is disjoint from  $bd(F_i)$ , the  $W_i$  are disjoint and the result follows.  $\square$

It is easy to prove that the face-vertex incidence graph of each  $G_i$  has radius  $O(k^2)$ . Hence each  $G_i$  has tree-width (defined below) which is  $O(k^2)$ . We show in Section 3 that we can find minimum odd cycle transversals in linear time in graphs with bounded tree-width. So if we could find all the  $G_i$ 's in linear time then we could compute a minimum odd cycle transversal for each  $G_i$  in linear time and by taking their union, find a minimum odd cycle transversal of  $G$  (or determine that  $G$  has no odd cycle transversal of order at most  $k$ ). This is close to what we do. There is one slight technical hitch, we actually need to consider a subgraph  $G''$  of  $G'$ . We give details in the next section.

We close this introductory section with some more remarks on related work concerning odd cycle packing and covering in planar graphs. Reed [11] showed that the following Erdős–Pósa property holds in planar graphs: for any integer  $k$ , there exists an  $f(k)$  such that  $G$  either has an odd cycle transversal of size at most  $f(k)$  or a packing of vertex disjoint odd cycles of size at least  $k + 1$ . For the edge version of this problem, Král and Voss [9] recently proved that  $f(k) = 2k$ . In contrast, it is easy to show that in general graphs the Erdős–Pósa property does not hold.

## 2. The algorithm

Our algorithm works as follows. First obtain an embedding of  $G$  in linear time [7], and construct the face-vertex incidence graph  $G^+$ . Then find a collection  $\mathcal{F} = \{f_1, \dots, f_s\}$  of vertex-disjoint odd faces of  $G$  which either has  $k + 1$  faces or is inclusion-wise maximal. We can do this in  $s + 1 \leq k + 2$  iterations in each of which we either add an odd face to our collection or determine that it is inclusion-wise maximal. This part of the algorithm can be implemented in  $O(kn)$  time which is linear as  $k$  is fixed.

If  $s > k$  then return the information that  $G$  has no odd cycle transversal of size at most  $k$  and stop. Otherwise, let  $B_i$  denote the set of faces and vertices of  $G$  whose distance to  $f_i$  in  $G^+$  is at most  $k + 3$ . Determine the sets  $B_i$  for all  $i = 1, \dots, s$  via a breadth first search in  $G^+$ . Let  $G''$  be the subgraph of  $G$  obtained by deleting all the vertices in each  $B_i$ .

Determine the set  $F_1, \dots, F_r$  of faces of the embedding of  $G''$  which contain an odd face of  $G$ . Note that  $r \leq s \leq k$  as each  $F_i$  contains some  $f_j \in \mathcal{F}$ . We let  $D_i$  be the subgraph of  $G$  contained in the union of  $F_i$  and its boundary. We refer to these graphs as *discs*. Now find a minimum odd cycle transversal  $W_i$  in each disc  $D_i$ . Since, as we show below, each disc has bounded tree-width, this can be done in linear time using the techniques described in Section 3. Let  $W$  be the union of  $W_1, \dots, W_r$ . If  $W$  has size at most  $k$ , then  $W$  is a minimum odd cycle transversal of  $G$ ; output  $W$ . Otherwise, return the information that  $G$  has no odd cycle transversal of size at most  $k$ . This concludes the description of the algorithm. Its correctness follows immediately from the fact that  $v$  is at distance at least  $k + 3$  from every face in  $\mathcal{F}$ , then  $d_{\min}(v)$  is at least  $k + 1$  and therefore, by Corollary 1.2,  $v$  does not belong to any inclusionwise minimal odd cycle transversal.

**Proposition 2.1.** *The algorithm finds a minimum cardinality odd cycle transversal if  $G$  has an odd cycle transversal of size at most  $k$  or otherwise detects that no such transversal exists.*

**Proof.** The proof of this proposition mimics exactly the proof of Corollary 1.3 with  $G'$  replaced by  $G''$  and  $G_i$  replaced by  $D_i$ .  $\square$

In Section 3, we will describe how to find minimum odd cycle transversals in graphs of bounded tree-width in linear time. Since all of the steps described in this section can be carried out in linear time, Proposition 2.1 tells us that we will obtain a linear time algorithm for general planar graphs if we can show that each disc has bounded tree-width. This, though, follows simply from the following result.

**Lemma 2.2** (Alon et al. [1], for a more general result see Robertson et al. [14,15]). *If a planar graph contains no  $h \times h$  grid minor, then its tree-width is at most  $8h$ .*

Since the  $h \times h$  grid has a unique planar embedding and the radius of the face-vertex incidence graph of  $G$  does not increase when edges of the graph are deleted or contracted, we have that the radius of the face-vertex incidence graph of any planar graph containing a  $h \times h$  grid minor is at least  $h$ . Hence, the preceding lemma has the following corollary:

**Corollary 2.3.** *Let  $G$  be a planar graph. If the radius of the face-vertex incidence graph of  $G$  is less than  $h$ , then the tree-width of  $G$  is at most  $8h$ .*

**Lemma 2.4.** *The tree-width of each disc is  $O(k^2)$ .*

**Proof.** By Corollary 2.3, it suffices to show that the radius of each disc is  $O(k^2)$ . Consider any disc  $D_i$ . Let  $I$  be the set of indices  $\ell$  such that  $f_\ell \in \mathcal{F}$  is a face of  $D_i$ . Let  $H$  denote the graph whose vertex set is  $I$  and whose edges are the pairs  $\ell\ell'$  of indices such that some vertex of  $G$  in  $B_\ell$  and some vertex of  $G$  in  $B_{\ell'}$  are incident to some common face of  $G$ . We know that  $H$  is connected and has at most  $k$  vertices, so its radius is at most  $k/2$ . Let  $j$  be a vertex of  $H$  such that the distance in  $H$  between  $j$  and any vertex of  $H$  is at most  $k/2$ . The distance in  $D_i^+$  between any  $f_\ell$  with  $\ell \in I$  and  $f_j$  is at most the distance in  $H$  between  $\ell$  and  $j$  times  $2(k+3)+2=2k+8$ . Moreover, for every face or vertex of  $D_i$  there is an index  $\ell \in I$  such that the distance in  $D_i^+$  between the considered face or vertex of  $D_i$  and  $f_j$  is at most  $(k/2)(2k+8)+k+5=k^2+5k+5$ . So the radius of  $D_i^+$  is indeed  $O(k^2)$ .  $\square$

### 3. Odd cycle transversals in graphs of bounded tree-width

As we have seen, it suffices to find a linear time algorithm for graphs with bounded tree-width. We observe that the bipartization problem for a fixed  $K$  can be expressed by an MSOL formula and thus can be decided in linear time for graphs of bounded tree-width [4]. Moreover, this is also implied by a result of Arnborg et al. [2]. For completeness, though, we think it worthwhile to include an explicit description of a linear time algorithm in the paper.

We begin with the required technical definitions. A *tree-decomposition* of  $G$  is a pair  $(T, \mathcal{V})$ , where  $T$  is a tree and  $\mathcal{V} = (V_t \subseteq V(G) : t \in V(T))$  is a family of subsets of  $V(G)$  with the following properties:

- (1)  $\bigcup (V_t : t \in V(T)) = V(G)$ .
- (2) For each edge  $e \in E(G)$  there is a  $t \in V(T)$  such that both endpoints of  $e$  are in  $V_t$ .
- (3) For  $t_0, t_1$  and  $t_2$  in  $V(T)$ , if  $t_0$  is on the path of  $T$  between  $t_1$  and  $t_2$ , then  $V_{t_1} \cap V_{t_2} \subseteq V_{t_0}$ .

The *width* of the tree-decomposition  $(T, \mathcal{V})$  is defined as  $\max_{t \in V(T)} (|V_t| - 1)$ . The *tree-width* of a graph  $G$  is the minimum  $w$  such that  $G$  has a tree-decomposition of width  $w$ . It is well known that there are minimum tree-decompositions of  $G$  that use at most  $n$  nodes. Moreover, we can easily convert a tree-decomposition  $(T, \mathcal{V})$  to another  $(T', \mathcal{V}')$  of the same width, such that  $T'$  is a binary tree with at most twice as many nodes as  $T$ . Let  $G$  be a graph with bounded tree-width  $\omega - 1$  and let  $(T, \mathcal{V})$  be a binary minimum tree-decomposition of  $G$ . We denote by  $t$  the nodes of  $T$  and by  $V_t$  the subset of  $V(G)$  assigned to  $t$ . We have that  $|V_t| \leq \omega$  for all  $t \in T$ . Pick an arbitrary root node  $t^* \in T$ . Then, given a node  $t \in T$  we let  $S_t$  be the subtree of  $T$  rooted at  $t$ . From (2) we may assign to each edge  $e = (u, v)$  of  $G$  a specific node  $t(e) \in T$  for which  $u, v \in V_t$ . Thus, for each  $t \in T$  there is an associated edge set  $E_t \subseteq E(G)$ . Hence, we may define the graphs  $G(t) = (V_t, E_t)$  and  $G(S_t) = (\bigcup_{t' \in S_t} V_{t'}, \bigcup_{t' \in S_t} E_{t'})$ .

We associate with each node  $t \in T$  a set  $\mathcal{A}_t$  of all the ordered triplets  $\Pi_t = (L_t, R_t, W_t)$  where  $L_t, R_t$  and  $W_t$  form a vertex partition of  $V_t$ . Clearly  $|\mathcal{A}_t|$  is at most  $3^\omega$ . Our algorithm will work up from the leaves maintaining the property that for each partition  $\Pi_t$  we (implicitly) store a minimum odd cycle transversal  $\hat{W}_t$  in  $G(S_t)$  that is *accordant* with the partition. That is,  $W_t \subseteq \hat{W}_t$  and  $L_t$  and  $R_t$  are on opposites sides of the bipartition in  $G(S_t) - \hat{W}_t$ . If such a transversal exists then we will set  $f(\Pi_t) = |\hat{W}_t|$ ; otherwise if there is no such accordant transversal then we set  $f(\Pi_t) = \infty$ . Hence, for a leaf  $t \in T$  we have  $f(\Pi_t) = |W_t|$  if  $L_t$  and  $R_t$  both induce stable sets in  $E_t$ . Otherwise  $f(\Pi_t) = \infty$ . Now take a non-leaf node  $t \in T$  with children  $r$  and  $s$ . If  $L_t$  or  $R_t$  induce an edge in  $E_t$  then we set  $f(\Pi_t) = \infty$ . So suppose not. We say that a partition  $\Pi_r = (L_r, R_r, W_r)$  in  $\mathcal{A}_r$  is *consistent* with a partition  $\Pi_t = (L_t, R_t, W_t)$  in  $\mathcal{A}_t$  if  $W_t \cap V(S_r) \subseteq W_r$ ,  $L_t \cap V(S_r) \subseteq L_r$  and  $R_t \cap V(S_r) \subseteq R_r$ . We use the notation  $\Pi_r \sim \Pi_t$  to denote consistency. Note, by property (3), that if  $\Pi_r$  and  $\Pi_s$  are both consistent with  $\Pi_t$  then they are consistent with each other. Then set

$$f(\Pi_t) = \min_{\Pi_r \sim \Pi_t, \Pi_s \sim \Pi_t} f(\Pi_r) + f(\Pi_s) + |W_t - (W_r \cup W_s)| - |W_r \cap W_s|.$$

Note that it may still be the case that  $f(\Pi_t) = \infty$ . We repeat this process up the tree. Observe that, by storing pointers from a partition  $\Pi_t$  to the partitions  $\Pi'_r$  and  $\Pi'_s$  in its children that produced the minimum value  $f(\Pi_t)$ , we may implicitly store the set  $\hat{W}_t$ . We then obtain the following result.

**Lemma 3.1.** *For each  $\Pi_t$ , either  $f(\Pi_t)$  is the size of the minimum odd cycle transversal in  $G(S_t)$  accordant with the partition  $\Pi_t$ , or  $f(\Pi_t) = \infty$  and no such a transversal exists.*

**Proof.** This is clearly true if  $t$  is a leaf. So let  $t \in T$  be a non-leaf with children  $r$  and  $s$ . Take  $\Pi_t$  and assume first that  $f(\Pi_t)$  is finite. Next take consistent partitions  $\Pi_r$  and  $\Pi_s$  with optimal transversals  $\hat{W}_r$  and  $\hat{W}_s$ , respectively. Then, since  $\hat{W}_r$  and  $\hat{W}_s$  are accordant with  $\Pi_r$  and  $\Pi_s$ , by property (3) we have that  $W_t - (\hat{W}_r \cup \hat{W}_s) = W_t - (W_r \cup W_s)$ . Thus, in obtaining  $\hat{W}_t$  we only need to add the vertices in  $W_t - (W_r \cup W_s)$ . Moreover any vertex in  $W_r \cap W_s$  is double counted by  $f(\Pi_r) + f(\Pi_s)$ . Thus  $f(\Pi_t)$  is, in fact, the size of a transversal in  $G(t)$  accordant with  $\Pi_t$ . Therefore, since we are examining all consistent pairs of partitions for the children, it is clear that  $f(\Pi_t)$  is the size of a minimum odd cycle transversal  $\hat{W}_t$  in  $G(S_t)$  accordant with the partition  $\Pi_t$ . Now suppose  $f(\Pi_t) = \infty$  and that there is a transversal  $W$  for  $G(S_t)$  accordant with  $\Pi_t$ . Then, for all pairs of partitions  $\Pi_r$  and  $\Pi_s$  that are consistent with  $\Pi_t$ , at least one of  $f(\Pi_r)$  or  $f(\Pi_s)$  is infinite. We obtain a contradiction as the restrictions of  $W$  to  $G(S_r)$  and  $G(S_s)$  give odd cycle transversals for these subgraphs that are accordant with  $\Pi_r$  and  $\Pi_s$ , respectively.  $\square$

It immediately follows that the minimum transversal can be found by considering the partition  $\Pi_{t^*}$  with the minimum  $f$  value. We may obtain a binary tree-decomposition in linear time [3]. For each node in the tree we have  $O(3^\omega)$  partitions. It takes  $O(|E_t|)$  time to check whether  $L_t$  or  $R_t$  induce stable sets in  $G(t)$ . There are then  $O(9^\omega)$  possible pairs of partitions for the children. Thus it takes  $O(\omega 9^\omega)$  time to check for consistencies and to calculate  $f(\Pi_t)$ . In total, therefore the algorithm runs in time  $O(\omega 3^{3\omega} n)$ . Thus we have proven Theorem 3.2 and Corollary 3.3.

**Theorem 3.2.** *Let  $G$  be a graph with bounded tree-width. Then there is an linear time algorithm to find a minimum odd cycle transversal in  $G$ .*

**Corollary 3.3.** *In a planar graph, for any constant  $k$ , there is an  $O(n)$  time algorithm to find a minimum odd cycle transversal of cardinality at most  $k$  or determine that no such transversal exists.*

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